INDICE

Páginas

INTEGRALES INMEDIATAS

Calcular $\int (3x+4)^2\,dx$

Si $u = 3x+4 \quad \mapsto \quad du = 3\,dx \quad \mapsto \quad dx = 1/3\,du$

$$\int (3x+4)^2\,dx = \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}(3x+4)^3 + C$$

Calcular $\int \frac{x}{x^2-1}dx$

$$\int \frac{x}{x^2-1}dx = \frac{1}{2}\int \frac{2x}{x^2-1}dx = \frac{1}{2}L(x^2-1) + C = L\sqrt{x^2-1} + C$$

Calcular $\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}dx$

$$d\cos\sqrt{x} = -\frac{1}{2\sqrt{x}}\operatorname{sen}\sqrt{x}\,dx \quad \mapsto \quad -2\,d\cos\sqrt{x} = \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}dx$$

$$\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}dx = -2\int d\cos\sqrt{x} = -2\cos\sqrt{x} + C$$

Calcular $\int \frac{dx}{9+x^2}$

$$\int \frac{dx}{9+x^2} = \frac{1}{9}\int \frac{dx}{1+(x/3)^2} = \frac{1}{3}\int \frac{(1/3)\,dx}{1+(x/3)^2} = \frac{1}{3}\operatorname{arc\,tg}\frac{x}{3} + C$$

Calcular $\int \frac{Lx}{x}dx$

Si $u = Lx \quad \mapsto \quad du = \frac{dx}{x}$

$$\int \frac{Lx}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{(Lx)^2}{2} + C$$

Calcular $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

$$\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-(x/2)^2}} = \int \frac{(1/2)\,dx}{\sqrt{1-(x/2)^2}} = \text{arc tg}\,\frac{x}{2} + C$$

Calcular $\displaystyle\int \frac{e^{1/x^2}\,dx}{x^3}$

$$u = \frac{1}{x^2} \quad \mapsto \quad du = -\frac{2}{x^3}\,dx \quad \mapsto \quad \frac{dx}{x^3} = -\frac{1}{2}\,du$$

$$\int \frac{e^{1/x^2}\,dx}{x^3} = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{1/x^2} + C$$

INTEGRACIÓN POR PARTES

Calcular $\int \cos x \; e^{-x} \, dx$

$u = e^{-x} \quad \Rightarrow \quad du = -e^{-x}\,dx$
$dv = \cos x\,dx \quad \Rightarrow \quad v = \operatorname{sen} x$

$$\int \cos x \; e^{-x}\,dx = \operatorname{sen} x \; e^{-x} + \int \operatorname{sen} x \; e^{-x}\,dx \quad \bullet$$

$$\left.\begin{array}{l} u = e^{-x} \quad \Rightarrow \quad du = -e^{-x}\,dx \\ dv = \operatorname{sen} x\,dx \quad \Rightarrow \quad v = -\cos x \end{array}\right\} \int \operatorname{sen} x \; e^{-x}\,dx = -\cos x \; e^{-x} - \int \cos x \; e^{-x}\,dx$$

$$\bullet \quad \int \cos x \; e^{-x}\,dx = \operatorname{sen} x \; e^{-x} - \cos x \; e^{-x} - \int \cos x \; e^{-x}\,dx$$

$$2\int \cos x \; e^{-x}\,dx = \operatorname{sen} x \; e^{-x} - \cos x \; e^{-x} \quad \mapsto \quad \int \cos x \; e^{-x}\,dx = \frac{e^{-x}}{2}(\operatorname{sen} x - \cos x) + C$$

Calcular $\int Lx \, dx$

$$\left.\begin{array}{l} u = Lx \quad \Rightarrow \quad du = \dfrac{dx}{x} \\ dv = dx \quad \Rightarrow \quad v = x \end{array}\right\} \int Lx\,dx = x\,Lx - \int dx = x\,Lx - x + C$$

Calcular $\int L(a^2 + x^2)\,dx$

$$\left.\begin{array}{l} u = L(a^2+x^2) \quad \Rightarrow \quad du = \dfrac{2x}{a^2+x^2}\,dx \\ dv = dx \quad \Rightarrow \quad v = x \end{array}\right\} \int L(a^2+x^2)\,dx = x\,L(a^2+x^2) - \int \frac{2x^2}{a^2+x^2}\,dx \quad \oplus$$

$$\int \frac{2x^2}{a^2+x^2}\,dx = \int \left(2 - \frac{2a^2}{a^2+x^2}\right) dx = 2x - 2\int \frac{a^2}{a^2+x^2}\,dx = 2x - 2\int \frac{dx}{1+(x/a)^2} = 2x - 2a \operatorname{arc\,tg}\frac{x}{a}$$

$$\oplus \quad \int L(a^2+x^2)\,dx = x\,L(a^2+x^2) - 2x + 2a \operatorname{arc\,tg}\frac{x}{a} + C$$

Calcular $\displaystyle\int \text{arc tg}\,x\,dx$

$$\left\{\begin{array}{l} u = \text{arc tg}\,x \;\Rightarrow\; du = \dfrac{dx}{1+x^2} \\ v = dx \;\Rightarrow\; v = x \end{array}\right\} \int \text{arc tg}\,x\,dx = x\,\text{arc tg}\,x - \int \frac{x\,dx}{1+x^2} = x\,\text{arc tg}\,x - \frac{1}{2}\,L(1+x^2) + C$$

Calcular $\displaystyle\int \frac{x\,\text{arc sen}\,x}{\sqrt{(1-x^2)^3}}\,dx$

$$\left.\begin{array}{l} u = \text{arc sen}\,x \;\Rightarrow\; du = \dfrac{dx}{\sqrt{1-x^2}} \\ dv = x\,(1-x^2)^{-3/2}\,dx \;\Rightarrow\; v = (1-x^2)^{-1/2} \end{array}\right\}$$

$$\int \frac{x\,\text{arc sen}\,x}{\sqrt{(1-x^2)^3}}\,dx = \frac{1}{\sqrt{1-x^2}}\,\text{arc sen}\,x - \int \frac{dx}{1-x^2} = \frac{\text{arc sen}\,x}{\sqrt{1-x^2}} - \text{ArgTh}\,x + C$$

Adviértase $\text{ArgTh}\,x = \dfrac{1}{2}\,L\dfrac{1+x}{1-x} = L\sqrt{\dfrac{1+x}{1-x}}$

Calcular $\displaystyle\int \frac{Lx}{(1-x)^{3/2}}\,dx$

$$\left.\begin{array}{l} u = Lx \;\Rightarrow\; du = \dfrac{dx}{x} \\ dv = (1-x)^{-3/2}\,dx \;\Rightarrow\; v = 2\,(1-x)^{-1/2} \end{array}\right\} \int \frac{Lx}{(1-x)^{3/2}}\,dx = \frac{2\,Lx}{\sqrt{1-x}} - 2\int \frac{dx}{x\,\sqrt{1-x}} \quad \oplus$$

$$\left.\begin{array}{l} 1-x = u^2 \quad dx = -2u\,du \\ x = 1-u^2 \end{array}\right\} \int \frac{dx}{x\,\sqrt{1-x}} = -2\int \frac{du}{1-u^2} = -2\,\text{ArgTh}\sqrt{1-x} = -L\left|\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right|$$

$$\oplus \quad \int \frac{Lx}{(1-x)^{3/2}}\,dx = \frac{2\,Lx}{\sqrt{1-x}} - 2\int \frac{dx}{x\,\sqrt{1-x}} = \frac{2\,Lx}{\sqrt{1-x}} + 2\,L\left|\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right| + C$$

Calcular $\int \operatorname{arc\,cotg} x \, dx$

$$u = \operatorname{arc\,cotg} x \Rightarrow du = \frac{-dx}{1+x^2}$$
$$dv = dx \Rightarrow v = x$$

$$\int \operatorname{arc\,cotg} x \, dx = x \operatorname{arc\,cotg} x + \int \frac{x\,dx}{1+x^2} = x \operatorname{arc\,cotg} x + \frac{1}{2} L(1+x^2) + C$$

Calcular $\int x\sqrt{4-5x}\, dx$

$$u = x \Rightarrow du = dx$$
$$dv = \sqrt{4-5x}\, dx \Rightarrow v = \int \sqrt{4-5x}\, dx = -\frac{2}{15}(4-5x)^{3/2}$$

$$\int \sqrt{4-5x}\, dx = \int (4-5x)^{1/2}\, dx = \begin{cases} t = 4-5x \\ dt = -5dx \end{cases} = \frac{-1}{5}\int t^{1/2}\, dt = -\frac{2}{15}(4-5x)^{3/2}$$

$$\int x\sqrt{4-5x}\, dx = -\frac{2}{15} x\,(4-5x)^{3/2} + \frac{2}{15}\int (4-5x)^{3/2} dx \quad \oplus$$

$$\int (4-5x)^{3/2}\, dx = \begin{cases} t = 4-5x \\ dt = -5dx \end{cases} = \frac{-1}{5}\int t^{3/2}\, dt = -\frac{2}{25}(4-5x)^{5/2}$$

$$\oplus \quad \int x\sqrt{4-5x}\, dx = -\frac{2}{15} x\,(4-5x)^{3/2} - \frac{4}{375}(4-5x)^{5/2} + C$$

Calcular $\int x^3\, e^{-x^2}\, dx$

$$u = x^2 \Rightarrow du = 2x\, dx$$
$$dv = x\, e^{-x^2}\, dx \Rightarrow v = -\frac{1}{2} e^{-x^2}$$

$$\int x^3\, e^{-x^2}\, dx = -\frac{1}{2} x^2\, e^{-x^2} + \int x\, e^{-x^2} dx = -\frac{1}{2} x^2\, e^{-x^2} - \frac{1}{2} e^{-x^2} + C$$

Calcular $\int e^{2x}\,\text{sen}\,x\,dx$

$$\left.\begin{array}{l} u = e^{2x} \;\Rightarrow\; du = 2e^{2x}\,dx \\ dv = \text{sen}\,x\,dx \;\Rightarrow\; v = -\cos x \end{array}\right\} \int e^{2x}\,\text{sen}\,x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \quad \oplus$$

$$\left.\begin{array}{l} u = e^{2x} \;\Rightarrow\; du = 2e^{2x}\,dx \\ dv = \cos x\,dx \;\Rightarrow\; v = \text{sen}\,x \end{array}\right\} \int e^{2x}\cos x\,dx = e^{2x}\,\text{sen}\,x - 2\int e^{2x}\,\text{sen}\,x\,dx$$

$$\oplus \quad \int e^{2x}\,\text{sen}\,x\,dx = -e^{2x}\cos x + 2\,e^{2x}\,\text{sen}\,x - 4\int e^{2x}\,\text{sen}\,x\,dx$$

$$5\int e^{2x}\,\text{sen}\,x\,dx = -e^{2x}\cos x + 2\,e^{2x}\,\text{sen}\,x \quad \mapsto \quad \int e^{2x}\,\text{sen}\,x\,dx = \frac{e^{2x}}{5}(2\,\text{sen}\,x - \cos x) + C$$

CÁLCULO INTEGRAL: FUNCIONES RACIONALES

Calcular $\displaystyle\int \frac{x+1}{x^2-4x+8}\,dx$

El numerador es de un grado menos que el denominador, se trata de un logaritmo

$$\int \frac{x+1}{x^2-4x+8}\,dx = \frac{1}{2}\int \frac{2x+2+4-4}{x^2-4x+8}dx = \frac{1}{2}\int \frac{2x-4}{x^2-4x+8}dx + \frac{1}{2}\int \frac{2x+6}{x^2-4x+8}dx =$$

$$= \frac{1}{2}\int \frac{2x-4}{x^2-4x+8}dx + 3\int \frac{dx}{x^2-4x+8} = \frac{1}{2}L(x^2-4x+8)+3I \quad \bullet$$

$$I = \int \frac{dx}{x^2-4x+8} = \int \frac{dx}{x^2-4x+8} = \int \frac{dx}{4+(x-2)^2} = \frac{1}{4}\int \frac{dx}{1+[(x-2)/2]^2} =$$

$$= \frac{1}{2}\int \frac{1/2\,dx}{1+[(x-2)/2]^2} = \frac{1}{2}\,\text{arc tg}\frac{x-2}{2}$$

- Resulta $\displaystyle\int \frac{x+1}{x^2-4x+8}\,dx = \frac{1}{2}L(x^2-4x+8)+\frac{3}{2}\,\text{arc tg}\frac{x-2}{2}+C$

Calcular $\displaystyle\int \frac{x^2}{x^2+2x+1}\,dx$

Numerador y denominador son del mismo grado, se puede dividir la función:

$$\int \frac{x^2}{x^2+2x+1}\,dx = \int \left[1-\frac{2x+1}{x^2+2x+1}\right]dx = x - \int \frac{2x+1}{x^2+2x+1}\,dx \quad \bullet$$

La integral $\displaystyle\int \frac{2x+1}{x^2+2x+1}\,dx = \int \frac{2x+2}{x^2+2x+1}\,dx - \int \frac{dx}{x^2+2x+1} =$

$$= L(x^2+2x+1) - \int (x+1)^{-2}\,dx = L(x^2+2x+1) - \frac{(x+1)^{-1}}{-1}$$

- En definitiva: $\displaystyle\int \frac{x^2}{x^2+2x+1}\,dx = x - L(x^2+2x+1) - \frac{1}{x+1} + C$

Calcular $\displaystyle\int \frac{x^3 + x^2 - x + 1}{x^2+1}\,dx$

Se divide la función subintegral al se el numerador de mayor grado que el denominador

$$\int \frac{x^3 + x^2 - x + 1}{x^2+1}\,dx = \int\left(x+2-\frac{2x+1}{x^2+1}\right)dx = \int (x+2)\,dx - \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} =$$

$$= \frac{x^2}{2} + 2x - L(x^2+1) - \text{arc tg}\,x + C$$

Calcular $\displaystyle\int \frac{x^4}{(x^2+1)^2}\,dx$

a) Se trata de una función racional, que queda:

$$\int \frac{x^4}{(x^2+1)^2}\,dx = \int\left(1-\frac{2x^2+1}{(x^2+1)^2}\right)dx = x - \int \frac{2x^2+1}{(x^2+1)^2}\,dx \quad \bullet$$

Aplicando el método de Hermite:

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{d}{dx}\left[\frac{Cx+D}{x^2+1}\right]$$

Derivando e identificando coeficientes, resulta:

$$2x^2 + 1 = A\,x^3 + (B-C)\,x^2 + (A-2D)\,x + (B+C)$$

$$\begin{cases} A=0 \\ B-C=2 \\ A-2D=0 \\ B+C=1 \end{cases} \mapsto \begin{cases} A=0 \quad D=0 \\ B-C=2 \\ B+C=1 \end{cases} \mapsto \begin{matrix} A=0 & D=0 \\ B=\dfrac{3}{2} & C=\dfrac{-1}{2} \end{matrix}$$

$$\int \frac{2x^2+1}{(x^2+1)^2}\,dx = \frac{3}{2}\int \frac{dx}{x^2+1} - \frac{x}{2\,(x^2+1)} = -\frac{x}{2\,(x^2+1)} + \frac{3}{2}\text{arc tg}\,x$$

$$\bullet \int \frac{x^4}{(x^2+1)^2}\,dx = x - \int \frac{2x^2+1}{(x^2+1)^2}\,dx = \boxed{x + \frac{x}{2\,(x^2+1)} - \frac{3}{2}\text{arc tg}\,x + C}$$

b) $\displaystyle\int \frac{x^4}{(x^2+1)^2}\,dx$ se puede resolver mediante integración por partes:

$u = x^3 \qquad du = 3x^2 dx$

$$dv = \frac{x}{(x^2+1)^2}\,dx \qquad v = \int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{dz}{z^2} = -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^4}{(x^2+1)^2}\,dx = \frac{-x^3}{2(x^2+1)} + \frac{3}{2}\int \frac{x^2\,dx}{x^2+1} = \frac{-x^3}{2(x^2+1)} + \frac{3}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx =$$

$$= \frac{-x^3}{2(x^2+1)} + \frac{3}{2}x - \frac{3}{2}\int \frac{1}{1+x^2}\,dx = \boxed{\frac{-x^3}{2(x^2+1)} + \frac{3}{2}x - \frac{3}{2}\operatorname{arc\,tg} x + C}$$

Obsérvese, $\dfrac{-x^3}{2(x^2+1)} + \dfrac{3}{2}x = \dfrac{2x^3+3x}{2(x^2+1)} = x + \dfrac{x}{2(x^2+1)}$

Ejercicios — Calcular $\displaystyle\int \frac{x^4}{x^4-1}\,dx$

Se divide la función subintegral $\dfrac{x^4}{x^4-1}$:

$$\int \frac{x^4}{x^4-1}\,dx = \int \left(1 + \frac{1}{x^4-1}\right)dx = \int dx + \int \frac{1}{x^4-1}\,dx = x + \int \frac{1}{x^4-1}\,dx \quad \bullet$$

Para calcular $\displaystyle\int \frac{1}{x^4-1}\,dx$ se descompone el polinomio (x^4-1) por Ruffini:

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Operando e identificando coeficientes:

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$$\left.\begin{array}{ll} \text{si } x=1 & 1=4A \\ \text{si } x=-1 & 1=-4B \\ \text{si } x=i & 1=-2(Ci+D) \end{array}\right\} \mapsto \left\{\begin{array}{l} A=1/4 \quad B=-1/4 \\ 1=-2Ci-2D \;\rightarrow \begin{cases} C=0 \\ D=-1/2 \end{cases} \end{array}\right.$$

- En consecuencia, resulta:

$$\int \frac{x^4}{x^4-1}\,dx = \int\left(1+\frac{1}{x^4-1}\right)dx = x + \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x^2+1} =$$

$$= x + \frac{1}{4}L(x-1) - \frac{1}{4}L(x+1) - \frac{1}{2}\operatorname{arc\,tg} x + C = \boxed{x + L\sqrt[4]{\frac{x-1}{x+1}} - \frac{1}{2}\operatorname{arc\,tg} x + C}$$

CÁLCULO INTEGRAL: FUNCIONES IRRACIONALES

Calcular $\displaystyle\int \frac{x\,dx}{1+\sqrt{x}}$

Irracional simple: cambio $x = t^2 \;\mapsto\; \begin{cases} t = \sqrt{x} \\ dx = 2t\,dt \end{cases}$

$$\int \frac{x\,dx}{1+\sqrt{x}} = 2\int \frac{t^3\,dt}{1+t} = 2\int \left(t^2 - t + 1 - \frac{1}{1+t}\right) dt = = 2\left[\frac{\sqrt{x^3}}{3} - \frac{x}{2} + \sqrt{x} - L\left(1+\sqrt{x}\right)\right] + C$$

Calcular $\displaystyle\int \frac{dx}{\sqrt[4]{x^3} - \sqrt{x}}$

Irracional simple: cambio $x = t^4 \;\mapsto\; \begin{cases} t = \sqrt[4]{x} \\ dx = 4t^3\,dt \end{cases}$

$$\int \frac{dx}{\sqrt[4]{x^3} - \sqrt{x}} = 4\int \frac{t^3\,dt}{t^3 - t^2} = 4\int \frac{t\,dt}{t-1} = 4\int \left(1 + \frac{1}{t-1}\right) dt = 4\left[\sqrt[4]{x} + L\left(\sqrt[4]{x} - 1\right)\right] + C$$

Calcular $\displaystyle\int \sqrt{\frac{x+1}{x-1}}\,dx$

Irracional lineal: cambio $\dfrac{x+1}{x-1} = t^2 \quad x = \dfrac{t^2+1}{t^2-1} \quad dx = \dfrac{-4t}{(t^2-1)^2}dt$

$$\int \sqrt{\frac{x+1}{x-1}}\,dx = \int \frac{-4t^2}{(t^2-1)^2}\,dt \qquad \oplus$$

Considerando $d\left(\dfrac{1}{t^2-1}\right) = \dfrac{-2t\,dt}{(t^2-1)^2}$ se integra por partes:

$u = 2t \;\Rightarrow\; du = 2\,dt$

$dv = \dfrac{-2t}{(t^2-1)^2}\,dt \;\Rightarrow\; v = \displaystyle\int \frac{-2t}{(t^2-1)^2}\,dt = \int d\left(\frac{1}{t^2-1}\right) = \frac{1}{t^2-1}$

$$\oplus \quad \int \frac{-4t^2}{(t^2-1)^2}\,dt = \int 2t\left(\frac{-2t}{(t^2-1)^2}\,dt\right) = \frac{2t}{t^2-1} - \int \frac{2\,dt}{t^2-1} = \frac{2t}{t^2-1} + 2\int \frac{dt}{1-t^2} =$$

$$= \frac{2t}{t^2-1} + 2\,\text{ArgTh}t + C = \sqrt{x^2-1} + 2\,\text{ArgTh}\sqrt{\frac{x+1}{x-1}} + C$$

Operando: $\dfrac{2t}{t^2-1} = \dfrac{2\sqrt{\dfrac{x+1}{x-1}}}{\dfrac{x+1}{x-1}-1} = \dfrac{2\,(x-1)\,\sqrt{x+1}}{2\,\sqrt{x-1}} = \sqrt{x^2-1}$

Calcular $\displaystyle\int \frac{x^3\,dx}{\sqrt{a^2+x^2}}$

Irracional binomia tipo $\displaystyle\int x^k\,(b+a\,x^h)^p\,dx$ donde $k=3 \quad h=2 \quad p=-\dfrac{1}{2}$

Siendo $\dfrac{k+1}{h}=2$ entero, se hace el cambio $a^2+x^2=t^2 \begin{cases} x^2 = t^2 - a^2 \\ x\,dx = t\,dt \end{cases}$

$$\int \frac{x^3\,dx}{\sqrt{a^2+x^2}} = \int \frac{(t^2-a^2)\,t}{t}\,dt = \int (t^2-a^2)\,dt = \frac{t^3}{3} - a^2\,t + C = \frac{\sqrt{(a^2+x^2)^3}}{3} - a^2\sqrt{a^2+x^2} + C$$

Calcular $\displaystyle\int \frac{dx}{(9+4x^2)^{5/2}}$

$$\int \frac{dx}{(9+4x^2)^{5/2}} = \int (9+4x^2)^{-5/2}\,dx$$

Irracional binomia tipo $\displaystyle\int x^k\,(b+a\,x^h)^p\,dx$ donde $k=0 \quad h=2 \quad p=-\dfrac{5}{2}$

Donde $\dfrac{k+1}{h}+p=-2$ entero, se multiplica y divide por $x^{hp} \equiv x^{2(-5/2)} = x^{-5}$

$$\int (9+4x^2)^{-5/2}\,dx = \int x^{-5}\,\frac{(9+4x^2)^{-5/2}}{x^{2(-5/2)}}\,dx = \int x^{-5}\left(\frac{9+4x^2}{x^2}\right)^{-5/2} dx \quad \oplus$$

Se hace el cambio $\dfrac{9+4x^2}{x^2} = t^2 \;\mapsto\; \begin{cases} x^2 = 9\,(t^2-4)^{-1} \\ x = 3\,(t^2-4)^{-1/2} \end{cases} \;\mapsto\; dx = -3\,t\,(t^2-4)^{-3/2}\,dt$

$$\oplus \quad \int x^{-5}\left(\frac{9+4x^2}{x^2}\right)^{-5/2} dx = -\frac{1}{81}\int (t^2-4)^{5/2}\; t^{-5}\; t\; (t^2-4)^{-3/2}\,dt =$$

$$= -\frac{1}{81}\int t^{-4}\,(t^2-4)\,dt = -\frac{1}{81}\int (t^{-2}-4\,t^{-4})\,dt = -\frac{1}{81}\left[-\frac{1}{t}+\frac{4}{3\,t^3}\right] + C =$$

$$=\frac{1}{81}\frac{x}{\sqrt{9+4x^2}}-\frac{4}{243}\frac{x^3}{\sqrt{(9+4x^2)^3}}+C$$

Calcular $\displaystyle\int \frac{dx}{x\sqrt{x^2+4x-4}}$

Se trata de una integral del tipo $\displaystyle\int R\left[x,\sqrt{ax^2+bx+c}\right]dx$ que cuando $a>0$ se transforma en una integral racional con el cambio $\sqrt{ax^2+bx+c}=\sqrt{a}\,x+t$

$$\sqrt{x^2+4x-4}=x+t \quad \mapsto \quad \cancel{x^2}+4x-4=\cancel{x^2}+2xt+t^2 \quad \mapsto \quad x=\frac{4+t^2}{4-2t}$$

$$t=\sqrt{x^2+4x-4}-x$$

$$dx=\frac{2(-t^2+4t+4)}{(-2t+4)^2}dt$$

$$\sqrt{x^2+4x-4}=\frac{4+t^2}{4-2t}+t=\frac{-t^2+4t+4}{4-2t}$$

$$\int\frac{dx}{x\sqrt{x^2+4x-4}}=\int\frac{\cancel{4-2t}}{4+t^2}\,\frac{\cancel{4-2t}}{\cancel{-t^2+4t+4}}\,\frac{\cancel{2(-t^2+4t+4)}}{\cancel{(-2t+4)^2}}dt=2\int\frac{dt}{4+t^2}=\frac{2}{4}\int\frac{dt}{1+(t/2)^2}=$$

$$=\int\frac{(1/2)\,dt}{1+(t/2)^2}=\operatorname{arctg}\frac{t}{2}+C=\boxed{\operatorname{arctg}\frac{\sqrt{x^2+4x-4}-x}{2}+C}$$

Calcular $\displaystyle\int \frac{dx}{(x+1)\sqrt{-4x^2+5x+9}}$

Se trata de una integral del tipo $\displaystyle\int R\left[x,\sqrt{ax^2+bx+c}\right]dx$ que cuando $\begin{cases}a<0\\ c>0\end{cases}$ se transforma en una integral racional con el cambio $\sqrt{ax^2+bx+c}=t\,x+\sqrt{c}$

$$\sqrt{-4x^2+5x+9}=t\,x+3 \quad \mapsto \quad \begin{cases}-4x^2+5x+\cancel{9}=t^2x^2+6tx+\cancel{9}\\ \quad -4x+5=t^2x+6t\end{cases} \quad \mapsto \quad x=\frac{5-6t}{4+t^2}$$

$$(x+1)=\frac{t^2-6t+9}{4+t^2} \qquad \sqrt{-4x^2+5x+9}=t\left(\frac{5-6t}{4+t^2}\right)+3=\frac{-3t^2+5t+12}{4+t^2}$$

$$x=\frac{5-6t}{4+t^2} \;\mapsto\; dx=\frac{2(3t^2-5t-12)}{(4+t^2)^2}\,dt$$

$$\int \frac{dx}{(x+1)\sqrt{-4x^2+5x+9}}=\int \frac{\cancel{4+t^2}}{(t^2-6t+9)}\times\frac{\cancel{4+t^2}}{\cancel{(-3t^2+5t+12)}}\times\frac{2\cancel{(3t^2-5t-12)}}{(\cancel{4+t^2})^2}\,dt=$$

$$=-2\int \frac{dt}{t^2-6t+9}=-2\int\frac{dt}{(t-3)^2}=\frac{2}{t-3}+C=\boxed{\frac{2x}{\sqrt{-4x^2+5x+9}-3(1+x)}+C}$$

Siendo $\sqrt{-4x^2+5x+9}=tx+3 \;\mapsto\; t=\dfrac{\sqrt{-4x^2+5x+9}-3}{x}$

Calcular $\displaystyle\int \frac{dx}{(x-2)\sqrt{-x^2+5x-4}}$

1 MÉTODO

Se trata de una integral del tipo $\displaystyle\int \frac{dx}{(h+kx)^n\sqrt{ax^2+bx+c}}$ que con el cambio $\dfrac{1}{h+kx}=t$ se transforma en una integral $\displaystyle\int\frac{P(x)\,dx}{\sqrt{ax^2+bx+c}}$ siendo $P(x)$ un polinomio de grado n.

$$\frac{1}{x-2}=t \;\mapsto\; x=2+\frac{1}{t} \;\mapsto\; dx=-\frac{1}{t^2}dt$$

$$\sqrt{-x^2+5x-4}=\sqrt{-\left(2+\frac{1}{t}\right)^2+5\left(2+\frac{1}{t}\right)-4}=\sqrt{\frac{2t^2+t-1}{t^2}}=\frac{\sqrt{2t^2+t-1}}{t}$$

$$\int\frac{dx}{(x-2)\sqrt{-x^2+5x-4}}=-\int t\,\frac{t}{\sqrt{2t^2+t-1}}\,t^{-2}\,dt=-\int\frac{dt}{\sqrt{2t^2+t-1}}=\quad\odot$$

Completando el cuadrado de $(2t^2+t-1)$:

$$2t^2+t-1=\left(\sqrt{2}\,t+\frac{1}{2\sqrt{2}}\right)^2-\frac{9}{8}=\left(\frac{4t+1}{2\sqrt{2}}\right)^2-\frac{9}{8}=\frac{1}{8}(4t+1)^2-\frac{9}{8}=\frac{9}{8}\left[\left(\frac{4t+1}{3}\right)^2-1\right]$$

$$\odot=-\frac{2\sqrt{2}}{3}\int\frac{dt}{\sqrt{\left(\frac{4t+1}{3}\right)^2-1}}=-\frac{2\sqrt{2}}{3}\int\frac{dt}{\sqrt{\left(\frac{4t+1}{3}\right)^2-1}}=-\frac{\sqrt{2}}{2}\int\frac{dt}{\sqrt{\left(\frac{4t+1}{3}\right)^2-1}}=$$

$$= -\frac{\sqrt{2}}{2}\ \text{Arg Ch}\left(\frac{4t+1}{3}\right) + C$$

Deshaciendo el cambio $t = \dfrac{1}{x-2}$ resulta:

$$\int \frac{dx}{(x-2)\sqrt{-x^2+5x-4}} = -\frac{\sqrt{2}}{2}\ \text{Arg Ch}\left(\frac{2+x}{3x-6}\right) + C$$

2 MÉTODO

Se trata de una integral del tipo $\int R\left[x, \sqrt{ax^2+bx+c}\right] dx$ que cuando $\begin{Bmatrix} a<0 \\ c<0 \end{Bmatrix}$ se transforma en una integral racional con el cambio $\sqrt{ax^2+bx+c} = t(x-x_1)$ siendo x_1 una raíz de $ax^2+bx+c=0$

Siendo $(-x^2+5x-4) = -(x-4)(x-1)$ se puede hacer el cambio:

$$\sqrt{-x^2+5x-4} = \sqrt{-(x-4)(x-1)} = t(x-1) \quad \mapsto \quad \begin{cases} -(x-4)(x-1) = t^2(x-1)^2 \\ x = \dfrac{4+t^2}{1+t^2} \end{cases}$$

Calcular $\int \sqrt{4x^2+3x-5}\ dx$

Es una integral del tipo $\int \sqrt{ax^2+bx+c}\ dx$ que multiplando y dividiendo por $\sqrt{ax^2+bx+c}\ dx$ se transforma en una integral $\int \dfrac{P(x)\,dx}{\sqrt{ax^2+bx+c}}$ siendo P(x) un polinomio de grado n, que se resuelve planteando la igualdad:

$$\int \frac{P(x)\,dx}{\sqrt{ax^2+bx+c}} = Q(x)\,.\sqrt{ax^2+bx+c} + K\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Q(x) es un polinomio de grado (n - 1). Para calcular las constantes hay que derivar ambos miembros de la igualdad.

$$\int \sqrt{4x^2+3x-5}\ dx = \int \frac{4x^2+3x-5}{\sqrt{4x^2+3x-5}}dx = (Cx+D)\,.\sqrt{4x^2+3x-5} + E\int \frac{dx}{\sqrt{4x^2+3x-5}}$$

Derivando la expresión:

$$\frac{4x^2+3x-5}{\sqrt{4x^2+3x-5}} = C\,\sqrt{4x^2+3x-5} + (Cx+D)\,\frac{8x+3}{2\sqrt{4x^2+3x-5}} + \frac{E}{\sqrt{4x^2+3x-5}}$$

$$4x^2+3x-5 = C\,(4x^2+3x-5) + (Cx+D)\left(\frac{4x+3}{2}\right) + E$$

Identificando coeficientes: $\begin{cases} 8C = 4 \\ \dfrac{9}{2}\,C + 4D = 3 \\ -5C + \dfrac{3}{2}\,D + E = -5 \end{cases} \quad \mapsto \quad \begin{cases} C = 1/2 \\ D = 3/16 \\ E = -89/32 \end{cases}$

Resultando:

$$\int \sqrt{4x^2+3x-5}\,dx = \left(\frac{1}{2}x + \frac{3}{16}\right).\sqrt{4x^2+3x-5} - \frac{89}{32}\int \frac{dx}{\sqrt{4x^2+3x-5}} \qquad \odot$$

Completando el cuadrado

$$4x^2+3x-5 = \left(2x+\frac{3}{4}\right)^2 - \frac{89}{16} = \left(\frac{8x+3}{4}\right)^2 - \frac{89}{16} = \frac{89}{16}\left[\left(\frac{8x+3}{\sqrt{89}}\right)^2 - 1\right]$$

con lo que,

$$\frac{89}{32}\int \frac{dx}{\sqrt{4x^2+3x-5}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\dfrac{8x+3}{\sqrt{89}}\right)^2 - 1}} = \frac{1}{2}\int \frac{(\sqrt{89}/8)\,du}{\sqrt{u^2-1}} = \frac{\sqrt{89}}{16}\,\text{Arg Ch}\left(\frac{8x+3}{\sqrt{89}}\right)$$

$$\odot \quad \int \sqrt{4x^2+3x-5}\,dx = \left(\frac{1}{2}x + \frac{3}{16}\right).\sqrt{4x^2+3x-5} - \frac{\sqrt{89}}{16}\,\text{Arg Ch}\left(\frac{8x+3}{\sqrt{89}}\right) + C$$

CÁLCULO INTEGRAL: MÉTODO DE HERMITE

Calcular $\displaystyle\int \frac{2x-1}{(x^2-6x+13)^2}\,dx$

Aplicando el método de Hermite:

$$\frac{2x-1}{(x^2-6x+13)^2}=\frac{Ax+B}{x^2-6x+13}+\frac{d}{dx}\left[\frac{Cx+D}{x^2-6x+13}\right] \qquad \text{(i)}$$

efectuando la derivación, se tiene:

$$\frac{2x-1}{(x^2-6x+13)^2}=\frac{Ax+B}{x^2-6x+13}+\frac{C(x^2-6x+13)-(Cx+D)(2x-6)}{(x^2-6x+13)^2}$$

simplificando,

$$2x-1=(Ax+B)(x^2-6x+13)+C(x^2-6x+13)-(Cx+D)(2x-6)$$

$$2x-1=Ax^3+x^2(-6A+B-C)+x(13A-6B-2D)+(13B+13C+6D)$$

Identificando coeficientes:

$$\begin{cases}\underline{A=0}\\ -6A+B-C=0\\ 13A-6B-2D=2\\ 13B+13C+6D=-1\end{cases} \mapsto \begin{cases}B-C=0\\ -6B-2D=2\\ 13B+13C+6D=-1\end{cases} \mapsto \begin{cases}B=C\\ -6B-2D=2\\ 26B+6D=-1\end{cases}$$

$$A=0 \qquad B=C=\frac{5}{8} \qquad D=-\frac{23}{8}$$

En consecuencia:

$$\text{(i)} \int \frac{2x-1}{(x^2-6x+13)^2}dx=\int\frac{5/8}{x^2-6x+13}dx+\int\frac{d}{dx}\left[\frac{(5/8)x-(23/8)}{x^2-6x+13}\right]dx=$$

$$=\frac{5x-23}{8(x^2-6x+13)}+\frac{5}{8}\int\frac{dx}{x^2-6x+13}$$

Completando el cuadrado: $(x^2-6x+1)=4+(x-3)^2$

$$\int\frac{dx}{x^2-6x+13}dx=\int\frac{dx}{4+(x-3)^2}=\int\frac{dx}{1+[(x-3)/2]^2}=\frac{1}{2}\text{arc tag}\left(\frac{x-3}{2}\right)$$

Finalmente, $\displaystyle\int\frac{2x-1}{(x^2-6x+13)^2}dx=\frac{5x-23}{8(x^2-6x+13)}+\frac{5}{16}\text{arc tag}\left(\frac{x-3}{2}\right)+C$

Calcular $\displaystyle\int \frac{x+3}{x\,(x^2+x+1)^2}\,dx$

Aplicando el método de Hermite:

$$\frac{x+3}{x\,(x^2+x+1)^2} = \frac{A}{x} + \frac{B\,x+C}{x^2+x+1} + \frac{d}{dx}\left[\frac{Dx+E}{x^2+x+1}\right] \qquad \text{(i)}$$

efectuando la derivación, se tiene:

$$\frac{x+3}{x\,(x^2+x+1)^2} = \frac{A}{x} + \frac{B\,x+C}{x^2+x+1} + \frac{D(x^2+x+1)-(D\,x+E)(2\,x+1)}{(x^2+x+1)}$$

simplificando

$$x+3 = A(x^2+x+1)^2 + (B\,x+C)\,x\,(x^2+x+1) + D(x^2+x+1) - (D\,x+E)(2\,x+1)$$

$$x+3 = (A+B)\,x^4 + (2\,A+B+C)\,x^3 + (3\,A+B+C-D)\,x^2 + (2A+C-2E)\,x + (A+D-E)$$

Identificando coeficientes:

$$\begin{cases} A+B=0 \\ 2\,A+B+C=0 \\ 3\,A+B+C-D=0 \\ 2A+C-2E=1 \\ A+D-E=3 \end{cases} \mapsto \begin{cases} \underline{B=-A} \\ A+C=0 \\ 2\,A+C-D=0 \\ 2A+C-2E=1 \\ A+D-E=3 \end{cases} \mapsto \begin{cases} \underline{C=-A} \\ A-D=0 \\ A-2E=1 \\ A+D-E=3 \end{cases} \mapsto \begin{cases} \underline{D=A} \\ A-2E=1 \\ 2\,A-E=3 \end{cases}$$

$$A=\frac{5}{3} \qquad B=-\frac{5}{3} \qquad C=-\frac{5}{3} \qquad D=\frac{5}{3} \qquad E=\frac{1}{3}$$

En consecuencia:

$$\text{(i)} \int \frac{x+3}{x\,(x^2+x+1)^2}\,dx = \int \frac{5/3}{x}\,dx + \int \frac{-(5/3)\,x-(5/3)}{x^2+x+1}\,dx + \int \frac{d}{dx}\left[\frac{(5/3)\,x+(1/3)}{x^2+x+1}\right]dx =$$

$$= \frac{5}{3}\int \frac{dx}{x} - \frac{5}{3}\int \frac{x+1}{x^2+x+1}\,dx + \frac{5\,x+1}{3(x^2+x+1)} = \frac{5}{3}\,Lx + \frac{5\,x+1}{3(x^2+x+1)} - \frac{5}{3}\int \frac{x+1}{x^2+x+1}\,dx$$

- $\displaystyle\int \frac{x+1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{(2x+1)+1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{(2x+1)}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx =$

$$= \frac{1}{2}\,L(x^2+x+1) + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx = \sqrt{L(x^2+x+1)} + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx \qquad \text{(ii)}$$

Completando el cuadrado:

$$(x^2+x+1)=\frac{3}{4}+\left(x+\frac{1}{2}\right)^2=\frac{3}{4}\left[1+\left(\frac{2x+1}{\sqrt{3}}\right)^2\right]$$

$$\bullet\bullet\ \int\frac{1}{x^2+x+1}dx=\frac{4}{3}\int\frac{dx}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}=\frac{2\sqrt{3}}{3}\operatorname{arc\,tg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Finalmente,

$$\int\frac{x+3}{x\,(x^2+x+1)^2}dx=\frac{5}{3}Lx+\frac{5x+1}{3(x^2+x+1)}-\frac{5}{3}\sqrt{L(x^2+x+1)}-\frac{5}{3}\,\frac{1}{2}\,\frac{2\sqrt{3}}{3}\operatorname{arc\,tg}\left(\frac{2x+1}{\sqrt{3}}\right)+C=$$

$$=\frac{5}{3}Lx+\frac{5x+1}{3(x^2+x+1)}-\frac{5}{3}\sqrt{L(x^2+x+1)}-\frac{5}{3\sqrt{3}}\operatorname{arc\,tg}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

Calcular $\displaystyle\int\frac{dx}{(1+x^2)^3}$

Aplicando el método de Hermite:

$$\frac{1}{(1+x^2)^3}=\frac{Ax+B}{1+x^2}+\frac{d}{dx}\left[\frac{Cx^3+Dx^2+Ex+F}{(1+x^2)^2}\right]\qquad\text{(i)}$$

efectuando la derivación, se tiene:

$$\frac{1}{(1+x^2)^3}=\frac{Ax+B}{1+x^2}+\frac{(3Cx^2+2Dx+E)(1+x^2)-4(Cx^3+Dx^2+Ex+F)}{(1+x^2)^3}$$

simplificando

$$1=(Ax+B)(1+x^2)^2+(3Cx^2+2Dx+E)(1+x^2)-4(Cx^3+Dx^2+Ex+F)$$

$$1=Ax^5+(B-C)\,x^4+(2A-2D)\,x^3+(2B+3C-3E)\,x^2+(A+2D-4F)\,x+(B+E)$$

Identificando coeficientes:

$$\begin{cases}A=0\\B-C=0\\2A-2D=0\\2B+3C-3E=0\\A+2D-4F=0\\B+E=1\end{cases}\mapsto\begin{cases}A=D=F=0\quad B=C\\5B-3E=0\\B+E=1\end{cases}\mapsto\begin{cases}A=D=F=0\\B=C=3/8\\E=5/8\end{cases}$$

En consecuencia:

$$\text{(i)} \int \frac{dx}{(1+x^2)^3} = \int \frac{(3/8)\,dx}{1+x^2} + \int \frac{d}{dx}\left[\frac{(3/8)\,x^3 + (5/8)\,x}{(1+x^2)^2}\right] dx$$

$$\int \frac{dx}{(1+x^2)^3} = \frac{3}{8}\int \frac{dx}{1+x^2} + \frac{3x^3+5x}{8(1+x^2)^2} = \frac{3}{8}\operatorname{arc\,tg} x + \frac{3x^3+5x}{8(1+x^2)^2} + C$$

INTEGRALES TRIGONOMÉTRICAS

Calcular $\displaystyle\int \frac{dx}{1+\cos x}$

Se puede hacer el cambio $\left\{\begin{array}{l} t = tg\dfrac{x}{2} \\ \cos x = \dfrac{1-t^2}{1+t^2} \\ dx = \dfrac{2}{1+t^2}\,dt \end{array}\right.$

$1+\cos x = \dfrac{2}{1+t^2}$

$$\int \frac{dx}{1+\cos x} = \int \frac{(2/1+t^2)\,dt}{(2/1+t^2)} = \int dt = t + C = tg\frac{x}{2} + C$$

- También se puede considerar: $1+\cos x = 2\cos^2\dfrac{x}{2}$

Adviértase,

$$\cos x = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos\frac{x}{2}\,.\,\cos\frac{x}{2} - sen\frac{x}{2}\,.\,sen\frac{x}{2} = \cos^2\frac{x}{2} - sen^2\frac{x}{2}$$

$$\cos x = \cos^2\frac{x}{2} - sen^2\frac{x}{2} = \cos^2\frac{x}{2} - \left(1-\cos^2\frac{x}{2}\right) = 2\cos^2\frac{x}{2} - 1 \quad \mapsto \quad 1+\cos x = 2\cos^2\frac{x}{2}$$

Por tanto, $\displaystyle\int \frac{dx}{1+\cos x} = \int \frac{(1/2)\,dx}{\cos^2(x/2)} = tg(x/2) + C$

Calcular $\displaystyle\int \frac{dx}{\cos x}$

Haciendo el cambio: $\quad t = tg\dfrac{x}{2} \quad \cos x = \dfrac{1-t^2}{1+t^2} \quad dx = \dfrac{2}{1+t^2}\,dt$

$$\int \frac{dx}{\cos x} = \int \frac{(2/\,1+t^2)\,dt}{(1-t^2/\,1+t^2)} = 2\int \frac{dt}{1-t^2} \quad \bullet$$

$$\frac{1}{1-t^2} = \frac{1}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$$

Identificando coeficientes:

$$1 = (A-B)\,t + (A+B) \quad \mapsto \quad \left\{\begin{array}{l} A-B=0 \\ A+B=1 \end{array}\right. \quad \Rightarrow \quad A = B = \frac{1}{2}$$

- $2\int \frac{1}{1-t^2}\,dt = 2\int \frac{1/2}{1-t}dt + 2\int \frac{1/2}{1+t}dt = -\int \frac{-dt}{1-t} + \int \frac{dt}{1+t} = -L(1-t) + L(1+t) + C =$

$$= L\frac{1+t}{1-t} + C = \boxed{L\frac{1+tg\,x/2}{1-tg\,x/2} + C}$$

⊙ Considerando como impar en $\color{red}{cos\,x}$ se puede hacer el cambio:

$$senx = t \quad \mapsto \quad \begin{cases} cos\,x = \sqrt{1-sen^2 x} = \sqrt{1-t^2} \\ cosx\,dx = dt \quad \Rightarrow \quad dx = \dfrac{dt}{\sqrt{1-t^2}} \end{cases}$$

$$\int \frac{dx}{cosx} = \int \frac{dt}{1-t^2} = L\frac{1+t}{1-t} + C = \boxed{L\frac{1+tg\,x/2}{1-tg\,x/2} + C}$$

Calcular $\int cos^3 x\,dx$

$$\int cos^3 x\,dx = \int cos^2 x\,cos\,x\,dx = \int (1-sen^2 x)\,cos\,x\,dx = \int cos\,x\,dx - \int sen^2 x\,cos\,x\,dx =$$

$$= senx - \frac{1}{3}sen^3 x + C$$

Calcular $\int sen^4 x\,dx$

Siendo, $cos\,2x = cos(x+x) = cos^2 x - sen^2 x = \begin{cases} = cos^2 x - (1-cos^2 x) = 2cos^2 x - 1 \\ = (1-sen^2 x) - sen^2 x = 1 - 2sen^2 x \end{cases}$

$$cos\,2x = \begin{cases} = 2cos^2 x - 1 \\ = 1 - 2sen^2 x \end{cases} \quad \mapsto \quad \begin{array}{l} cos^2 x = \dfrac{1+cos\,2x}{2} \\ sen^2 x = \dfrac{1-cos\,2x}{2} \end{array}$$

$$\int sen^4 x\,dx = \int \left(\frac{1-cos\,2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2cos\,2x + cos^2 2x)\,dx =$$

$$= \frac{1}{4}x - \frac{1}{4}sen\,2x + \frac{1}{4}\int cos^2 2x\,dx = \frac{1}{4}x - \frac{1}{4}sen\,2x + \frac{1}{4}\int \left(\frac{1+cos\,4x}{2}\right) dx =$$

$$= \frac{1}{4}x - \frac{1}{4}\operatorname{sen}2x + \frac{1}{8}\int \cancel{dx} + \frac{1}{8}\int \cos 4x\,dx = \frac{1}{4}x - \frac{1}{4}\operatorname{sen}2x + \frac{1}{8}x + \frac{1}{32}\int 4\cos 4x\,dx =$$

$$= \frac{1}{4}x - \frac{1}{4}\operatorname{sen}2x + \frac{1}{8}x + \frac{1}{32}\operatorname{sen}4x + C$$

Calcular $\displaystyle\int \frac{dx}{\operatorname{sen}x\cos x}$

$$\int \frac{dx}{\operatorname{sen}x\cos x} = \int \frac{\operatorname{sen}^2 x + \cos^2 x}{\operatorname{sen}x\cos x}dx = \int \frac{\operatorname{sen}^2 x}{\operatorname{sen}x\cos x}dx + \int \frac{\cos^2 x}{\operatorname{sen}x\cos x}dx =$$

$$= \int \frac{\operatorname{sen}x}{\cos x}dx + \int \frac{\cos x}{\operatorname{sen}x}dx = -L\cos x + L\operatorname{sen}x + C = L\frac{\operatorname{sen}x}{\cos x} + C = L\operatorname{tg}x + C$$

Calcular $\displaystyle\int \frac{\operatorname{tg}x}{1+\operatorname{sen}^2 x}dx$

$$t = \operatorname{tg}x \quad x = \operatorname{arc\,tg}t \quad dx = \frac{dt}{1+t^2} \qquad t^2 = \frac{\operatorname{sen}^2 x}{\cos^2 x} = \frac{\operatorname{sen}^2 x}{1-\operatorname{sen}^2 x} \quad \mapsto \quad \operatorname{sen}x = \frac{t}{\sqrt{1+t^2}}$$

$$1+\operatorname{sen}^2 x = 1 + \frac{t^2}{1+t^2} = \frac{1+2t^2}{1+t^2}$$

$$\int \frac{\operatorname{tg}x}{1+\operatorname{sen}^2 x}dx = \int \frac{t\,\cancel{(1+t^2)}}{(1+2t^2)}\,\frac{dt}{\cancel{1+t^2}} = \int \frac{t}{(1+2t^2)}dt = \frac{1}{4}\int \frac{4t}{1+2t^2}dt = \frac{1}{4}L(1+2t^2) + C =$$

$$= \frac{1}{4}L(1+2\operatorname{tg}^2 x) + C = L\sqrt[4]{1+2\operatorname{tg}^2 x} + C$$

Calcular $\displaystyle\int \frac{dx}{\cos x - \operatorname{sen}x}$

$$t = \operatorname{tg}\frac{x}{2} \quad x = 2\operatorname{arc\,tg}t \quad dx = \frac{2dt}{1+t^2} \quad \operatorname{sen}x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\cos x - \operatorname{sen}x = \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = \frac{1-t^2-2t}{1+t^2}$$

$$\int \frac{dx}{\cos x - \operatorname{sen}x} = \int \frac{\cancel{1+t^2}}{-t^2-2t+1}\,\frac{2dt}{\cancel{1+t^2}} = -2\int \frac{dt}{t^2+2t-1} = -2\int \frac{dt}{(t+1)^2-2} \quad \oplus$$

$$\frac{1}{(t+1)^2-2}=\frac{1}{(t+1-\sqrt{2})\,(t+1+\sqrt{2})}=\frac{A}{t+1-\sqrt{2}}+\frac{B}{t+1+\sqrt{2}}$$

Identificando coeficientes $\begin{cases} A+B=0 \\ (1+\sqrt{2})\,A+(1-\sqrt{2})\,B=1 \end{cases} \mapsto \quad A=\frac{1}{2\sqrt{2}} \quad B=-\frac{1}{2\sqrt{2}}$

$$\oplus \quad -2\int\frac{dt}{(t+1)^2-2}=-\frac{1}{\sqrt{2}}\int\frac{dt}{t+1-\sqrt{2}}+\frac{1}{\sqrt{2}}\int\frac{dt}{t+1+\sqrt{2}}=$$

$$=-\frac{1}{\sqrt{2}}\,L\left|t+1-\sqrt{2}\right|+\frac{1}{\sqrt{2}}\,L\left|t+1+\sqrt{2}\right|+C=\frac{1}{\sqrt{2}}\,L\left|\frac{t+1+\sqrt{2}}{t+1-\sqrt{2}}\right|+C=\frac{1}{\sqrt{2}}\,L\left|\frac{tg\frac{x}{2}+1+\sqrt{2}}{tg\frac{x}{2}+1-\sqrt{2}}\right|+C$$

Calcular $\int sen^5x\,cos^2x\,dx$

$$\int sen^5x\,cos^2x\,dx=\int senx\,sen^4x\,cos^2x\,dx=\int senx\,(1-cos^2x)^2\,cos^2x\,dx=$$

$$=\int senx\,(1-2cos^2x+cos^4x)\,cos^2x\,dx=$$

$$=\int senx\,cos^2x\,dx-2\int senx\,cos^4x\,dx+\int senx\,cos^6x\,dx=$$

$$=-\frac{1}{3}cos^3x+\frac{2}{5}cos^5x-\frac{1}{7}cos^7x+C$$

Calcular $\int sen5x\,cos4x\,dx$

$$\left.\begin{array}{l} sen(5x+4x)=sen5x\,cos4x+\cancel{cos5x\,sen4x} \\ sen(5x-4x)=sen5x\,cos4x-\cancel{cos5x\,sen4x} \\ \hline sen9x+senx=2\,sen5x\,cos4x \end{array}\right\} \quad sen5x\,cos4x=\frac{1}{2}(sen9x+senx)$$

$$\int sen5x\,cos4x\,dx=\frac{1}{2}\int(sen9x+senx)\,dx=-\frac{1}{18}cos9x-\frac{1}{2}cosx+C$$

Calcular $\int tg^3 x\, dx$

$$u = tg\, x \quad x = arc\, tg\, u \quad dx = \frac{du}{1+u^2}$$

$$\int tg^3 x\, dx = \int \frac{u^3}{1+u^2} du = \int \left(u - \frac{u}{1+u^2} \right) du = \frac{u^2}{2} - \frac{1}{2} L(1+u^2) + C = \frac{tg^2 x}{2} - \frac{1}{2} L(1+tg^2 x) + C =$$

$$= \frac{tg^2 x}{2} - \frac{1}{2} L \frac{1}{\cos^2 x} + C = \frac{tg^2 x}{2} + \frac{1}{2} L \cos^2 x + C = \frac{tg^2 x}{2} + L \cos x + C$$

INTEGRALES FUNCIONES HIPERBÓLICAS

Calcular $\int \text{Sh}5x\ \text{Sh}3x\, dx$

El producto de la función subintegral se convierte en una suma:

$$\begin{aligned}
&\text{Ch}(5x+3x) = \text{Ch}5x\,.\,\text{Ch}3x + \text{Sh}5x\,.\,\text{Sh}3x \\
&\underline{\text{Ch}(5x-3x) = \text{Ch}5x\,.\,\text{Ch}3x - \text{Sh}5x\,.\,\text{Sh}3x} \\
&\text{Ch}\,8x - \text{Ch}\,2x = 2\,\text{Sh}5x\,.\,\text{Sh}3x
\end{aligned}$$

$$\int \text{Sh}5x\ \text{Sh}3x\, dx = \frac{1}{2}\int (\text{Ch}\,8x - \text{Ch}\,2x)\, dx \ = \frac{1}{16}\,\text{Sh}8x \ - \ \frac{1}{4}\,\text{Sh}2x \ + \ C$$

Calcular $\int \frac{\text{Sh}\,x}{1+\text{Sh}\,x}\, dx$

Se parte del cambio general de las integrales hiperbólicas:

$$t = \text{Th}\frac{x}{2} \quad \mapsto \quad dx = \frac{2\,dt}{1-t^2} \qquad \text{Sh}\,x = \frac{2t}{1-t^2} \qquad \text{Ch}\,x = \frac{1+t^2}{1-t^2} \qquad 1+\text{Sh}\,x = \frac{-t^2+2t+1}{1-t^2}$$

$$\int \frac{\text{Sh}\,x}{1+\text{Sh}\,x}\, dx = \int \frac{\cancel{1-t^2}}{-t^2+2t+1} \times \frac{2t}{\cancel{1-t^2}} \times \frac{2\,dt}{1-t^2} = \int \frac{4t\,dt}{(-t^2+2t+1)(1-t^2)} \qquad \odot$$

Descomponiendo en fracciones simples la función racional:

$$\frac{4t}{(-t^2+2t+1)(1-t^2)} = \frac{A+Bt}{-t^2+2t+1} + \frac{C}{1-t} + \frac{D}{1+t}$$

Identificando coeficientes:

$$4t = (-B-C+D)t^3 \ + \ (-A+C-3D)t^2 \ + \ (B+3C+D)t \ + \ (A+C+D)$$

$$A = -2 \quad B = 0 \quad C = D = 1$$

$$\odot \quad \int \frac{4t\,dt}{(-t^2+2t+1)(1-t^2)} = \int \frac{-2\,dt}{-t^2+2t+1} + \int \frac{dt}{1-t} + \int \frac{dt}{1+t} =$$

$$= -2\int \frac{dt}{2-(t-1)^2} + \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = -\sqrt{2}\int \frac{(1/\sqrt{2})dt}{1-\left(\frac{t-1}{\sqrt{2}}\right)^2} + \int \frac{dt}{1-t} + \int \frac{dt}{1+t} =$$

$$= -\sqrt{2}\ \text{Arg Th}\left(\frac{t-1}{\sqrt{2}}\right) - L|1-t| + L|1+t| + C = -\sqrt{2}\ \text{Arg Th}\left(\frac{t-1}{\sqrt{2}}\right) \ + \ L\left|\frac{1+t}{1-t}\right| \ + C$$

Deshaciendo el cambio $t = \text{Th}\frac{x}{2}$

$$\int \frac{\text{Sh}\,x}{1+\text{Sh}\,x}\,dx = -\sqrt{2}\ \text{Arg Th}\left(\frac{\text{Th}\frac{x}{2}-1}{\sqrt{2}}\right) + L\left|\frac{1+\text{Th}\frac{x}{2}}{1-\text{Th}\frac{x}{2}}\right| + C$$

CÁLCULO INTEGRAL: FUNCIÓN GAMMA

Calcular $\displaystyle\int_0^1 (Lx)^4\,dx$

Se lleva a una integral $\Gamma(p)=\displaystyle\int_0^\infty x^{p-1}e^{-x}\,dx$ con el cambio:

$$Lx=-t \quad\Rightarrow\quad \begin{cases} x=e^{-t} & \Rightarrow\quad dx=-e^{-t}\,dt \\ x=1 & \Rightarrow\quad t=0 \\ x=0 & \Rightarrow\quad t=\infty \end{cases}$$

$$\int_0^1 (Lx)^4\,dx=\int_\infty^0 -(-t)^4\,e^{-t}\,dt=\int_0^\infty (-t)^4\,e^{-t}\,dt=\int_0^\infty t^4\,e^{-t}\,dt=\Gamma(5)=4!=24 \qquad p-1=4\ \mapsto\ p=5$$

Calcular $\displaystyle\int_0^1 \sqrt[3]{L(1/x)}\,dx$

Se hace el cambio

$$L\frac{1}{x}=-L\,x=t \quad\Rightarrow\quad \begin{cases} x=e^{-t} & \Rightarrow\quad dx=-e^{-t}\,dt \\ x=1 & \Rightarrow\quad t=0 \\ x=0 & \Rightarrow\quad t=\infty \end{cases}$$

$$\int_0^1 \sqrt[3]{L(1/x)}\,dx=\int_\infty^0 -\sqrt[3]{t}\ e^{-t}\,dt=\int_0^\infty t^{1/3}\,e^{-t}\,dt=\Gamma\left(\frac{1}{3}+1\right)=\Gamma\left(\frac{4}{3}\right)$$

Calcular $\displaystyle\int_0^\infty x^4\,e^{-5x^2}\,dx$

Se hace el cambio

$$5x^2=t\ \mapsto\ x=\frac{t^{1/2}}{\sqrt{5}} \quad\Rightarrow\quad \begin{cases} dx=\dfrac{1}{2\sqrt{5}}\,t^{-1/2}\,dt \\ x=\infty \quad\Rightarrow\quad t=\infty \\ x=0 \quad\Rightarrow\quad t=0 \end{cases}$$

$$\int_0^\infty x^4\,e^{-5x^2}\,dx=\int_0^\infty \left(\frac{t^{1/2}}{\sqrt{5}}\right)^4\left(\frac{1}{2\sqrt{5}}\right)t^{-1/2}\ e^{-t}\ dt=\frac{1}{50\sqrt{5}}\int_0^\infty t^{3/2}\,e^{-t}\,dt=\frac{1}{50\sqrt{5}}\,\Gamma\left(\frac{3}{2}+1\right)$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)=\frac{3}{4}\sqrt{\pi}$$

Por tanto,

$$\int_0^{\infty} x^4\, e^{-5x^2}\, dx = \frac{1}{50\sqrt{5}}\Gamma\left(\frac{5}{2}\right) = \frac{3}{200}\sqrt{\frac{\pi}{5}}$$

CÁLCULO INTEGRAL: FUNCIÓN BETA

Calcular $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$

$$\int_0^1 \sqrt{\frac{1-x}{x}}\,dx = \int_0^1 x^{-1/2}\,(1-x)^{1/2}\,dx = \beta\left(\frac{1}{2},\frac{3}{2}\right) = \frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(2)} = \frac{\pi}{2}$$

Se considera $\begin{cases} p-1=-\frac{1}{2} & \mapsto \quad p=\frac{1}{2} \\ q-1=\frac{1}{2} & \mapsto \quad q=\frac{3}{2} \end{cases}$

Adviértase que $\Gamma(1/2)\Gamma(3/2) = \Gamma\left(\frac{1}{2}\right)\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,\frac{1}{2}\,\sqrt{\pi} = \frac{\pi}{2}$

Calcular $\int_0^1 \sqrt{1-x^5}\,dx$

Se hace el cambio: $x^5 = t \quad \mapsto \quad x = t^{1/5} \quad \Rightarrow \quad \begin{cases} dx = \frac{1}{5}t^{-4/5}\,dt \\ x=1 \Rightarrow t=1 \\ x=0 \Rightarrow t=0 \end{cases}$

$$\int_0^1 \sqrt{1-x^5}\,dx = \frac{1}{5}\int_0^1 t^{-4/5}\,(1-t)^{1/2}\,dt = \frac{1}{5}\beta\left(\frac{1}{5},\frac{3}{2}\right) = \frac{1}{5}\,\frac{\Gamma(1/5)\,\Gamma(3/2)}{\Gamma(17/10)}$$

Se considera $\begin{cases} p-1=-\frac{4}{5} & \mapsto \quad p=\frac{1}{5} \\ q-1=\frac{1}{2} & \mapsto \quad q=\frac{3}{2} \end{cases}$

Por recurrencia: $\Gamma(6/5) = \left(\frac{6}{5}-1\right)\Gamma\left(\frac{6}{5}-1\right) = \left(\frac{1}{5}\right)\Gamma\left(\frac{1}{5}\right)$

Resultando: $\int_0^1 \sqrt{1-x^5}\,dx = \frac{\Gamma(6/5)\,\Gamma(3/2)}{\Gamma(17/10)}$

Calcular $\displaystyle\int_0^{\infty} \frac{dx}{1+x^3}$

Se hace el cambio: $x^3 = t \quad \mapsto \quad x = t^{1/3} \quad \Rightarrow \quad \begin{cases} dx = \dfrac{1}{3} t^{-2/3}\, dt \\ x = \infty \;\Rightarrow\; t = \infty \\ x = 0 \;\Rightarrow\; t = 0 \end{cases}$

Resultando:

$$\int_0^{\infty} \frac{dx}{1+x^3} = \frac{1}{3}\int_0^{\infty} \frac{t^{-2/3}}{1+t}\, dt = \frac{1}{3}\beta\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}\frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1)} = \frac{1}{3}\Gamma(1/3)\Gamma(2/3)$$

Se ha considerado $\begin{cases} p-1 = -\dfrac{2}{3} \quad \mapsto \quad p = \dfrac{1}{3} \\ p+q = 1 \quad \mapsto \quad q = \dfrac{2}{3} \end{cases}$

Calcular $\displaystyle\int_0^{2} (4-x^2)^{3/2}\, dx$

$$(4-x^2) = 4\left[1-\left(\frac{x}{2}\right)^2\right] \quad \mapsto \quad (4-x^2)^{3/2} = 4^{3/2}\left[1-\left(\frac{x}{2}\right)^2\right]^{3/2} = 8\left[1-\frac{x^2}{4}\right]^{3/2}$$

$$\int_0^{2} (4-x^2)^{3/2}\, dx = 8\int_0^{2}\left[1-\frac{x^2}{4}\right]^{3/2} dx$$

Se hace el cambio: $\dfrac{x^2}{4} = t \quad \mapsto \quad x = 2t^{1/2} \quad \Rightarrow \quad \begin{cases} dx = t^{-1/2}\, dt \\ x = 2 \;\Rightarrow\; t = 1 \\ x = 0 \;\Rightarrow\; t = 0 \end{cases}$

$$\int_0^{2} (4-x^2)^{3/2}\, dx = 8\int_0^{2}\left[1-\frac{x^2}{4}\right]^{3/2} dx = 8\int_0^{1} t^{-1/2}(1-t)^{3/2}\, dt = 8\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

donde $\begin{cases} p-1 = -\dfrac{1}{2} \quad \mapsto \quad p = \dfrac{1}{2} \\ q-1 = \dfrac{3}{2} \quad \mapsto \quad q = \dfrac{5}{2} \end{cases}$

$$\int_0^{2} (4-x^2)^{3/2}\, dx = 8\beta\left(\frac{1}{2}, \frac{5}{2}\right) = 8\frac{\Gamma(1/2)\,\Gamma(5/2)}{\Gamma(3)} = 4\Gamma(1/2)\,\Gamma(5/2) = 3\pi$$

$$\begin{cases} \Gamma(1/2) = \sqrt{\pi} \\ \Gamma(5/2) = (3/2)\,\Gamma(3/2) = (3/2)\,(1/2)\,\Gamma(1/2) = (3/4)\sqrt{\pi} \end{cases}$$

Calcular $\displaystyle\int_0^{\pi} \frac{\sqrt{\operatorname{sen} x}}{(5+3\cos x)^{3/2}}\,dx$

Función trigonométrica racional, haciendo el cambio:

$$t=\operatorname{tg}\frac{x}{2} \ \mapsto\ x=2\operatorname{arctg} t \begin{cases} dx=\dfrac{2\,dt}{1+t^2} \\ \operatorname{sen} x=\dfrac{2t}{1+t^2} \quad \cos x=\dfrac{1-t^2}{1+t^2} \\ x=\pi \ \mapsto\ t=\infty \\ x=0 \ \mapsto\ t=0 \end{cases}$$

operando en la función subintegral:

$$\frac{\sqrt{\operatorname{sen} x}}{(5+3\cos x)^{3/2}}=\frac{2^{1/2}\,t^{1/2}\,(1+t^2)^{-1/2}}{\left(5+3\left(\dfrac{1-t^2}{1+t^2}\right)\right)^{3/2}}=\frac{2^{1/2}\,t^{1/2}\,(1+t^2)^{-1/2}}{\left(\dfrac{8+2t^2}{1+t^2}\right)^{3/2}}=$$

$$=\frac{2^{1/2}\,t^{1/2}\,(1+t^2)^{-1/2}}{2^{3/2}\,(4+t^2)^{3/2}\,(1+t^2)^{-3/2}}=\frac{t^{1/2}\,(1+t^2)}{2\,(4+t^2)^{3/2}}$$

con lo que

$$\int_0^{\pi}\frac{\sqrt{\operatorname{sen} x}}{(5+3\cos x)^{3/2}}\,dx=\int_0^{\infty}\frac{t^{1/2}\,\cancel{(1+t^2)}}{\cancel{2}\,(4+t^2)^{3/2}}\,\frac{\cancel{2}\,dt}{\cancel{1+t^2}}=\int_0^{\infty}\frac{t^{1/2}\,dt}{(4+t^2)^{3/2}}=$$

$$=\int_0^{\infty}\frac{t^{1/2}\,dt}{4^{3/2}\,(1+t^2/4)^{3/2}}=\frac{1}{8}\int_0^{\infty}\frac{t^{1/2}\,dt}{(1+t^2/4)^{3/2}} \quad (\bullet)$$

haciendo el cambio

$$u=\frac{t^2}{4} \ \mapsto\ \begin{matrix} t^2=4u \\ t=2u^{1/2} \\ t^{1/2}=\sqrt{2}\,u^{1/4} \end{matrix} \quad \begin{cases} dt=u^{-1/2}\,du \\ t=\infty \ \rightarrow\ u=\infty \\ t=0 \ \rightarrow\ u=0 \end{cases}$$

$$(\bullet)\ =\frac{1}{8}\int_0^{\infty}\frac{t^{1/2}\,dt}{(1+t^2/4)^{3/2}}=\frac{1}{8}\int_0^{\infty}\frac{\sqrt{2}\,u^{1/4}}{(1+u)^{3/2}}\,u^{-1/2}\,du=$$

$$=\frac{\sqrt{2}}{8}\int_0^{\infty}\frac{u^{-1/4}}{(1+u)^{3/2}}\,du=\frac{\sqrt{2}}{8}\,\beta\!\left(\frac{3}{4},\,\frac{3}{4}\right) \quad (\bullet\bullet)$$

En la función $\beta(p,\,q)\begin{cases} p-1=-1/4 \ \mapsto\ p=3/4 \\ p+q=3/2 \ \mapsto\ q=3/4 \end{cases}$

$$(\bullet\bullet)\int_0^{\pi}\frac{\sqrt{\operatorname{sen}x}}{(5+3\cos x)^{3/2}}\,dx=\frac{\sqrt{2}}{8}\,\beta\left(\frac{3}{4},\frac{3}{4}\right)=\frac{\sqrt{2}}{8}\,\frac{\Gamma(3/4)\,\Gamma(3/4)}{\Gamma(3/2)}=$$

$$=\frac{\sqrt{2}}{8}\,\frac{[\Gamma(3/4)]^2}{(1/2)\,\Gamma(1/2)}=\frac{\sqrt{2}}{4}\,\frac{[\Gamma(3/4)]^2}{\Gamma(1/2)}=\frac{\sqrt{2}}{4}\,\frac{[\Gamma(3/4)]^2}{\sqrt{\pi}}=\frac{[\Gamma(3/4)]^2}{2\sqrt{2\pi}}$$

Calcular $\displaystyle\int_0^{\pi/2}\frac{dx}{\sqrt{\operatorname{tg}x}}$

$$\int_0^{\pi/2}\frac{dx}{\sqrt{\operatorname{tg}x}}=\int_0^{\pi/2}\frac{dx}{\sqrt{\operatorname{sen}x/\cos x}}=\int_0^{\pi/2}(\operatorname{sen}x)^{-1/2}\,(\cos x)^{1/2}\,dx=$$

$$=\frac{1}{2}\int_0^{\pi/2}2\,(\operatorname{sen}x)^{-1/2}\,(\cos x)^{1/2}\,dx=\frac{1}{2}\,\beta\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

En la función $\beta(p,q)\begin{cases}2p-1=-1/2 & \mapsto\quad p=1/4\\ 2q-1=1/2 & \mapsto\quad q=3/4\end{cases}$ -

Calcular $\displaystyle\int_0^{2\pi}\operatorname{sen}^8x\,dx$

Como se trata de una potencia par de sen x, se tiene:

$$\int_0^{2\pi}\operatorname{sen}^8x\,dx=2\left(\int_0^{\pi/2}\operatorname{sen}^8x\,dx\right)=2\,\beta\left(\frac{9}{2},\frac{1}{2}\right)\qquad(\bullet)$$

En la función $\beta(p,q)\begin{cases}2p-1=8 & \mapsto\quad p=9/2\\ 2q-1=0 & \mapsto\quad q=1/2\end{cases}$ -

Por recurrencia: $\displaystyle\beta\left(\frac{9}{2}\right)=\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)=\frac{105}{16}\sqrt{\pi}$

$$2\,\beta\left(\frac{9}{2},\frac{1}{2}\right)=2\,\frac{\Gamma(9/2)\,\Gamma(1/2)}{\Gamma(5)}=2\,\frac{\frac{105}{16}\sqrt{\pi}\,\sqrt{\pi}}{4!}=\frac{35}{64}\,\pi$$

$$(\bullet)\int_0^{2\pi}\operatorname{sen}^8x\,dx=\frac{35}{64}\,\pi$$

CÁLCULO INTEGRAL: DERIVACIÓN DE UNA INTEGRAL PARAMÉTRICA

Calcular $\displaystyle\int_0^{\infty} \frac{\operatorname{sen} x}{x}\,dx$

Se parte de la integral $\displaystyle I = \int_0^{\infty} e^{-ax}\,\frac{\operatorname{sen} x}{x}\,dx \quad \otimes$

Siendo $\operatorname{Im}(e^{ix}) = \operatorname{Im}(\cos x + i \operatorname{sen} x) = \operatorname{sen} x$

derivando respecto al parámetro a

$$\frac{dI}{da} = \int_0^{\infty} -x\, e^{-ax}\,\frac{\operatorname{sen} x}{x}\,dx = \int_0^{\infty} -e^{-ax} \operatorname{sen} x\,dx = \operatorname{Im}\left(\int_0^{\infty} -e^{-ax}\, e^{ix}\,dx\right) =$$

$$= \operatorname{Im}\left(\int_0^{\infty} -e^{-(a-i)x}\,dx\right) = \operatorname{Im}\left(\frac{1}{a-i}\int_0^{\infty} -(a-i)\,e^{-(a-i)x}\,dx\right) =$$

$$= \operatorname{Im}\left(\frac{1}{a-i}\int_0^{\infty} -(a-i)\,e^{-(a-i)x}\,dx\right) = \operatorname{Im}\left(\frac{1}{a-i}\,e^{-(a-i)x}\right)_0^{\infty} = \operatorname{Im}\left(-\frac{1}{a-i}\right) =$$

$$= -\operatorname{Im}\left(\frac{1}{a-i}\right) = -\operatorname{Im}\left[\frac{a+i}{(a-i)\,(a+i)}\right] = -\operatorname{Im}\left[\frac{a+i}{1+a^2}\right] = \frac{-1}{1+a^2}$$

resultando

$$\frac{dI}{da} = \int_0^{\infty} -x\, e^{-ax}\,\frac{\operatorname{sen} x}{x}\,dx = \frac{-1}{1+a^2} \quad \mapsto \quad I = \int \frac{-1}{1+a^2}\,da = -\operatorname{arc\,tg} a + C$$

Para $a = \infty$ se tiene:

$$\otimes\ I_{a=\infty} = \int_0^{\infty} 0\,dx = 0 = -\operatorname{arc\,tag} \infty + C = -\frac{\pi}{2} + C \ \mapsto \ C = \frac{\pi}{2}$$

El cálculo de la integral pedida cuando $a = 0$:

$$\otimes \int_0^{\infty} \frac{\operatorname{sen} x}{x}\,dx = I_{a=0} = \int_0^{\infty} e^{-0}\,\frac{\operatorname{sen} x}{x}\,dx = -\operatorname{arc\,tg} 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

Calcular $\displaystyle\int_0^{\infty} \frac{\text{sen}^2 x}{x^2}\,dx$

Se parte de la integral $\displaystyle I = \int_0^{\infty} \frac{\text{sen}^2 ax}{x^2}\,dx \quad \otimes$

Derivando respecto al parámetro a

$$\frac{dI}{da} = \int_0^{\infty} \frac{2x\,\text{sen}ax\,\cos ax}{x^2}\,dx = \int_0^{\infty} \frac{2\,\text{sen}ax\,\cos ax}{x}\,dx = \int_0^{\infty} \frac{\text{sen}\,2ax}{x}\,dx \quad (\bullet)$$

haciendo el cambio: $2ax = t \;\Rightarrow \begin{cases} x = \dfrac{t}{2a} \quad dx = \dfrac{dt}{2a} \\ x = \infty \;\mapsto\; t = \infty \\ x = 0 \;\mapsto\; t = 0 \end{cases}$

$$(\bullet) \int_0^{\infty} \frac{\text{sen}\,2ax}{x}\,dx = \int_0^{\infty} \frac{\text{sen}\,t}{t}\,dt = \frac{\pi}{2}$$

En consecuencia, $\displaystyle \frac{dI}{da} = \frac{\pi}{2} \quad \mapsto \quad I = \int \frac{\pi}{2}\,da = \frac{\pi}{2}a + C$

Para a = 0, se tiene: $\displaystyle \otimes \; I_{a=0} = \int_0^{\infty} 0\,dx = 0 = 0 + C \;\mapsto\; C = 0$

Para el cálculo de la integral solicitada, se particulariza para a = 1

$$\otimes \; I_{a=1} = \int_0^{\infty} \frac{\text{sen}^2 x}{x^2}\,dx = \frac{\pi}{2}$$

Calcular $\displaystyle\int_0^1 x\,(Lx)^n\,dx$

Se parte de la integral $\displaystyle I=\int_0^1 x^a\,dx=\left.\frac{x^{a+1}}{a+1}\right|_0^1=\frac{1}{a+1}$

derivando sucesivamente respecto al parámetro a:

$$\frac{dI}{da}=\int_0^1 x^a\,Lx\,dx=\frac{d}{da}\left(\frac{1}{a+1}\right)=\frac{-1}{(a+1)^2}=\frac{-1!}{(a+1)^2}$$

$$\frac{d^2I}{da^2}=\int_0^1 x^a\,(Lx)^2\,dx=\frac{d}{da}\left[\frac{-1}{(a+1)^2}\right]=\frac{2.1}{(a+1)^3}=\frac{2!}{(a+1)^3}$$

$$\frac{d^3I}{da^3}=\int_0^1 x^a\,(Lx)^3\,dx=\frac{d}{da}\left[\frac{2}{(a+1)^3}\right]=\frac{-3.\,2.1}{(a+1)^4}=\frac{-3!}{(a+1)^4}$$

....

$$\frac{d^nI}{da^n}=\int_0^1 x^a\,(Lx)^n\,dx=\frac{d}{da}\left[\frac{2}{(a+1)^3}\right]=\frac{(-1)^n\,n!}{(a+1)^{n+1}}$$

Para $a=1$ se obtiene el cálculo solicitado:

$$\left.\frac{d^nI}{da^n}\right|_{a=1}=\left.\int_0^1 x^a\,(Lx)^n\,dx\right|_{a=1}=\int_0^1 x\,(Lx)^n\,dx=\left.\frac{(-1)^n\,n!}{(a+1)^{n+1}}\right|_{a=1}=\frac{(-1)^n\,n!}{2^{n+1}}$$

Calcular $\displaystyle\int_0^\infty L\left(1+\frac{a^2}{x^2}\right)dx$

Derivando respecto al parámetro a:

$$\frac{d}{da}\int_0^\infty L\left(1+\frac{a^2}{x^2}\right)dx=\int_0^\infty \frac{2a/x^2}{1+a^2/x^2}\,dx=2\int_0^\infty \frac{a/x^2}{1+(a/x)^2}\,dx\overset{\bullet}{=}-2\int_\infty^0 \frac{dt}{1+t^2}\,dt=$$

$$=2\int_0^\infty \frac{dt}{1+t^2}\,dt=2\,\text{arc tg}\,t=2\left[\text{arc tg}\frac{a}{x}\right]_0^\infty=2\left(\frac{\pi}{2}-0\right)=\pi \qquad \otimes$$

• cambio $t = \dfrac{a}{x} \mapsto \begin{cases} dt = -\dfrac{a}{x^2}\,dx \\ x = 0 \ \rightarrow \ t = \infty \\ x = \infty \ \rightarrow \ t = 0 \end{cases}$

$$\otimes \quad \frac{d}{da}\int_0^{\infty} L\left(1+\frac{a^2}{x^2}\right) dx = \pi \quad \Rightarrow \quad \int_0^{\infty} L\left(1+\frac{a^2}{x^2}\right) dx = \int \pi\, da = \pi a + C$$

Para $a = 0$: $\displaystyle\int_0^{\infty} L(1)\, dx = \int_0^{\infty} 0\, dx = 0 = 0 + C \ \rightarrow \ C = 0$

con lo que, $\displaystyle\int_0^{\infty} L\left(1+\frac{a^2}{x^2}\right) dx = \pi a$

Calcular $\displaystyle\int_0^{1} \frac{L(1+x)}{1+x^2}\, dx$

Se parte de la integral $\mathbf{I} = \displaystyle\int_0^{a} \frac{L(1+ax)}{1+x^2}\, dx$ donde $\begin{cases} f(x,a) = \dfrac{L(1+ax)}{1+x^2} \\ f(a,a) = \dfrac{L(1+a^2)}{1+a^2} \end{cases}$

Derivando respecto al parámetro **a**, teniendo en cuenta que el limite superior de integración es también función del parámetro.

$$\frac{d}{da}\int_0^{a} \frac{L(1+ax)}{1+x^2}\, dx = \int_0^{a} \frac{x}{(1+ax)\,(1+x^2)}\, dx + \frac{L(1+a^2)}{1+a^2} \quad \otimes$$

Resulta una integral racional en x, que se resuelva mediante descomposición en fracciones simples:

$$\frac{x}{(1+ax)\,(1+x^2)} = \frac{A}{1+ax} + \frac{Bx+C}{1+x^2} \quad \rightarrow \quad x = (A+aB)x^2 + (B+aC)x + (A+C)$$

Identificando coeficientes:

$$\begin{cases} A + C = 0 \\ B + aC = 1 \\ A + aB = 0 \end{cases} \left| \begin{array}{l} A = -C \\ B + aC = 1 \\ aB - C = 0 \end{array} \right. \quad \Rightarrow \quad B = \frac{1}{1+a^2} \quad C = \frac{a}{1+a^2} \quad A = \frac{-a}{1+a^2}$$

de donde,

$$\int_0^a \frac{x}{(1+ax)(1+x^2)}\,dx = \frac{-1}{1+a^2}\int_0^a \frac{a\,dx}{1+ax} + \frac{1}{1+a^2}\int_0^a \frac{(x+a)}{1+x^2}dx =$$

$$= \frac{-1}{1+a^2}\int_0^a \frac{a\,dx}{1+ax} + \frac{1}{1+a^2}\int_0^a \frac{x\,dx}{1+x^2} + \frac{a}{1+a^2}\int_0^a \frac{dx}{1+x^2} =$$

$$= \frac{-1}{1+a^2}\,L(1+ax)\Big|_0^a + \frac{1}{2(1+a^2)}\,L(1+x^2)\Big|_0^a + \frac{a}{1+a^2}\,\operatorname{arc\,tg} x\Big|_0^a =$$

$$= \frac{-1}{1+a^2}L(1+a^2) + \frac{1}{2(1+a^2)}L(1+a^2) + \frac{a}{1+a^2}\operatorname{arc\,tg} a =$$

$$= \frac{-L(1+a^2)}{1+a^2} + \frac{1}{2(1+a^2)}L(1+a^2) + \frac{a}{1+a^2}\operatorname{arc\,tg} a$$

$$\otimes\quad \frac{d}{da}\int_0^a \frac{L(1+ax)}{1+x^2}\,dx = \cancel{\frac{-L(1+a^2)}{1+a^2}} + \frac{1}{2(1+a^2)}L(1+a^2) + \frac{a}{1+a^2}\operatorname{arc\,tg} a + \cancel{\frac{L(1+a^2)}{1+a^2}}$$

$$\frac{d}{da}\int_0^a \frac{L(1+ax)}{1+x^2}\,dx = \frac{1}{2(1+a^2)}L(1+a^2) + \frac{a}{1+a^2}\operatorname{arc\,tg} a$$

En definitiva,

$$\int_0^a \frac{L(1+ax)}{1+x^2}\,dx = \int \left[\frac{1}{2(1+a^2)}L(1+a^2) + \frac{a}{1+a^2}\operatorname{arc\,tg} a\right] da =$$

$$= \int \frac{1}{2(1+a^2)}L(1+a^2)\;da + \int \frac{a}{1+a^2}\operatorname{arc\,tg} a\,da = \mathbf{I_1 + I_2} \quad \bullet\bullet$$

$$\mathbf{I_1} = \int \frac{1}{2(1+a^2)}L(1+a^2)\,da = \frac{1}{2}L(1+a^2)\operatorname{arc\,tg} a - \int \frac{a}{(1+a^2)}\operatorname{arc\,tg} a\,da$$

$$u = L(1+a^2) \qquad du = \frac{2a}{(1+a^2)}\,da$$

$$dv = \frac{da}{2(1+a^2)} \qquad v = \frac{1}{2}\operatorname{arc\,tg} a$$

$$\bullet\bullet\quad = \mathbf{I_1 + I_2} = \frac{1}{2}L(1+a^2)\operatorname{arc\,tg} a - \cancel{\int \frac{a}{(1+a^2)}\operatorname{arc\,tg} a\,da} + \cancel{\int \frac{a}{1+a^2}\operatorname{arc\,tg} a\,da} + C$$

$$\mathbf{I} = \int_0^a \frac{L(1+ax)}{1+x^2}\,dx = \frac{1}{2}\,L(1+a^2)\,\text{arc tg}\,a + C$$

Para $a=0$: $\mathbf{I}_{a=0} = \int_0^a 0\,dx = 0 = 0 + C \;\rightarrow C = 0$

Por tanto: $\mathbf{I} = \int_0^a \frac{L(1+ax)}{1+x^2}\,dx = \frac{1}{2}\,L(1+a^2)\,\text{arc tg}\,a$

Para $a=1$: $\mathbf{I}_{a=1} = \boxed{\int_0^1 \frac{L(1+x)}{1+x^2}\,dx = \frac{1}{2}\,L2\ \text{arc tg}\,1 = \frac{\pi}{4}\,L\sqrt{2}}$

Calcular $\int_0^\infty \frac{\text{arc tg}\,x}{x\,(1+x^2)}\,dx$

Se parte de la integral $I = \int_0^\infty \frac{\text{arc tg}\,ax}{x\,(1+x^2)}\,dx$

Derivando respecto al parámetro **a**

$$\frac{dI}{da} = \frac{d}{da}\int_0^\infty \frac{\not{x}}{\not{x}\,(1+x^2)\,(1+a^2x^2)}\,dx = \int_0^\infty \frac{dx}{(1+x^2)\,(1+a^2x^2)} \qquad \bullet$$

Resulta una integral racional en x, que se resuelva mediante descomposición en fracciones simples:

$$\frac{1}{(1+x^2)\,(1+a^2x^2)} = \frac{A\,x + B}{1+x^2} + \frac{C\,x + D}{1+a^2x^2}$$

Identificando coeficientes:

$$1 = (a^2A + C)\,x^3 + (a^2B + D)\,x^2 + (A + C)\,x + (B + D)$$

$$\begin{cases} a^2A + C = 0 \\ a^2B + D = 0 \\ A + C = 0 \\ B + D = 1 \end{cases} \rightarrow \begin{cases} C = -A \quad D = 1 - B \\ a^2A - A = 0 \\ a^2B + 1 - B = 0 \end{cases} \rightarrow \begin{cases} A = 0 \qquad C = 0 \\ B = \dfrac{-1}{a^2-1} \quad D = \dfrac{a^2}{a^2-1} \end{cases}$$

de donde,

- $$\frac{dI}{da}=\frac{d}{da}\int_0^{\infty}\frac{dx}{(1+x^2)\,(1+a^2x^2)}=\frac{-1}{a^2-1}\int_0^{\infty}\frac{dx}{1+x^2}+\frac{a}{a^2-1}\int_0^{\infty}\frac{a\,dx}{1+(ax)^2}=$$

$$=\frac{-1}{a^2-1}\,\text{arc tag}\,x\Big|_0^{\infty}+\frac{a}{a^2-1}\,\text{arc tag}\,ax\Big|_0^{\infty}=\frac{-\pi}{2(a^2-1)}+\frac{a\pi}{2(a^2-1)}=\frac{\pi}{2(a+1)}$$

$$\frac{dI}{da}=\int_0^{\infty}\frac{dx}{(1+x^2)\,(1+a^2x^2)}=\frac{\pi}{2(a+1)}\quad\mapsto\quad I=\frac{\pi}{2}\int\frac{da}{a+1}=\frac{\pi}{2}\,L(a+1)+C$$

La constante C se calcula particularizando para $a=0$:

$$I_{a=0}=\int_0^{\infty}0\,dx=0=\frac{\pi}{2}\,L(1)+C\quad\mapsto\quad C=0$$

Para calcular la integral solicitada se particulariza para $a=1$:

$$\boxed{I_{a=1}=\int_0^{\infty}\frac{\text{arc tg}\,x}{x\,(1+x^2)}\,dx=\frac{\pi}{2}\,L\,2}$$

Calcular $\displaystyle\int_{-\infty}^{\infty}e^{-ax^2+bx}\,dx$

Sea $\displaystyle I(a,b)=\int_{-\infty}^{\infty}e^{-ax^2+bx}\,dx$

Derivando respecto al parámetro **b**

$$I'_b=\frac{\vartheta I}{\vartheta b}=\frac{\vartheta}{\vartheta b}\int_{-\infty}^{\infty}e^{-ax^2+bx}\,dx=\int_{-\infty}^{\infty}x\,e^{-ax^2+bx}\,dx\quad\bullet$$

Haciendo arreglos para integrar con mayor comodidad:

- $$\int_{-\infty}^{\infty}x\,e^{-ax^2+bx}\,dx=\frac{-1}{2a}\int_{-\infty}^{\infty}-2ax\,e^{-ax^2+bx}\,dx=\frac{-1}{2a}\int_{-\infty}^{\infty}(-2ax+b-b)\,e^{-ax^2+bx}\,dx=$$

$$=\frac{-1}{2a}\int_{-\infty}^{\infty}(-2ax+b)\,e^{-ax^2+bx}\,dx+\frac{b}{2a}\int_{-\infty}^{\infty}e^{-ax^2+bx}\,dx\quad\bullet\bullet$$

siendo: $$\frac{-1}{2a}\int_{-\infty}^{\infty}(-2ax+b)\,e^{-ax^2+bx}\,dx=\frac{-1}{2a}\,e^{-ax^2+bx}\Big|_{-\infty}^{\infty}=0$$

$$\bullet\bullet\quad I'_b = \frac{\vartheta I}{\vartheta b} = \int_{-\infty}^{\infty} x\, e^{-ax^2+bx}\,dx = \frac{b}{2a}\int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = \frac{b}{2a}\, I$$

$$\frac{I'_b}{I} = \frac{b}{2a} \quad\to\quad \int \frac{I'_b}{I}\,db = \int \frac{b}{2a}\,db \;\Rightarrow\; \boxed{LI = \frac{b^2}{4a} + C}$$

Para calcular la constante C se particulariza para $b = 0$:

$$LI_{b=0} = L\int_{-\infty}^{\infty} e^{-ax^2}\,dx \overset{\otimes}{=} L\sqrt{\frac{\pi}{a}} = 0 + C \;\Rightarrow\; \underline{C = L\sqrt{\frac{\pi}{a}}}$$

$$\otimes \text{ donde} \int_{-\infty}^{\infty} e^{-ax^2}\,dx = 2\int_{0}^{\infty} e^{-ax^2}\,dx = \frac{1}{\sqrt{a}}\int_{0}^{\infty} t^{-1/2}\, e^{-t}\,dt = \frac{1}{\sqrt{a}}\,\Gamma(1/2) = \sqrt{\frac{\pi}{a}}$$

Por tanto, $LI = \dfrac{b^2}{4a} + L\sqrt{\dfrac{\pi}{a}} \;\mapsto\; LI = Le^{b^2/4a} + L\sqrt{\dfrac{\pi}{a}} \;\mapsto\; \boxed{I = \sqrt{\pi/a}\; e^{b^2/4a}}$